# MODELS OF MEASUREMENT UNCERTAINTY OPTIMIZATION IN DECISION-MAKING ON THE BASIS OF MEASUREMENT RESULTS

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*Abstract:* In this paper different approaches and associated models for measurement uncertainty optimization in decision making procedures are analyzed.

*Key-Words:* decision making, optimization, measurement uncertainty, hypothesis checking, loss function.

## Introduction

The requirements for measurement accuracy (uncertainty) are always set in accordance with the purpose of the measurement, which is determined by the further use of the measurement result. The measurement problem is part of a more general problem of decision making

This can be a purely metrological task, for example, verification of measuring instruments, comparison of calibration and testing laboratories, or it can arise in applied research such as ecology, medicine, trade, economics, etc. These tasks include the tasks of monitoring environmental parameters, quality products, management of technological processes, development of complex information and measurement systems, recognition of pollution sources, measurement in conflict situations (arbitration measurements), etc.

Quite often, the measuring problem can be separated into an independent task, and its connection with a more general problem is manifested only at the stage of planning the measurement, when the object of investigation, the measurand and the requirements for the measurement accuracy are determined. However, in a number of cases a separate solution of the measurement problem is impossible because the requirements for the measurement accuracy, as well as the decision-making criteria, depend on the measurand value. In such cases, adaptive decision-making procedures are required which are based on measurement results. At the time of these procedures the requirements for each measurement procedure can be specified, including the measurement accuracy, for example:

- When monitoring parameters characterizing the state of the object under study, the technological process, when a continuous chain of measurements and control actions is realized.
- In the development and study of so-called "test, control, determination, detection" methods that cannot be attributed to measurement methods, which cannot be related to methods of measurement, but which contain measurement procedures inside themselves.
- With metrological support of intelligent measuring instruments and information and measuring systems

Let's briefly stop on the main factors affecting the quality of decisions made and linked directly with the measuring procedure used:

- Accuracy of measurement information. Quantitatively, the accuracy of measurement information is characterized by measurement uncertainty. The statistical apparatus for evaluating the uncertainty has been developed quite well. It is important that a mathematic approach to evaluating the reliability of decisions made on the basis of measurement results has been consistent with it.
- Adequacy of measurement information. The adequacy of measurement information in this context is understood as the adequacy to the requirements for its further use, the "docking" of the measurement task and the decision-making tasks based on the measurement information. This requirement must be ensured at the stage of the measurement procedure development at the expense of correct (adequate, competent) selection of the measurand, measurement conditions, measuring instruments, parameters of the measurement experiment, data processing algorithm and algorithm for evaluating the accuracy of measurement results. This problem arises both in classical metrology (verification, calibration of measuring instruments, comparison of reference standards and interlaboratory experiments), and at the interface of classical and applied metrology: measurements

in monitoring environmental parameters and technological processes, measurements in ecology, medical measurements, measurements at commercial calculations when selling oil, gas, etc.

- Optimality. Taking into account the huge volume of measurements performed and the enormous material costs for their implementation, the process of obtaining measurement information should be optimized; including the uncertainty of measurements should be optimized too. Increasing the accuracy of measurements leads, on the one hand, to reducing the probabilities of erroneous decisions, and on the other hand, to an increase in the cost of measurements. In such a situation, the problem of optimizing measurement uncertainty arises.

## 1. Statement of the problem

Similarly to how the measurement result is not complete without indication of its uncertainty, so the decision-making procedure should be accompanied by evaluation of its quality and reliability. Thus, there is a range of issues / problems associated with ensuring the quality of decision-making procedures on the basis of measurement results:

- How to formalize the decision-making task on the basis of measurement results?
- What are the characteristics of the quality (reliability) of the decision-making procedure which have to be chosen?
- How to set a decision rule?
- How to evaluate the reliability of the decision-making process?
- How to ensure meeting the requirements for the reliability of the decision-making procedure?
- How to optimize the cost of decision-making at the given requirements for their reliability/

## 2. Mathematical formalism

A formalized approach to solving the problem of making decisions on the basis of measurement results is to apply the theory of statistical solutions. The tasks listed above to make decisions on the basis of measurement information in a formalized form can be represented as follows:

- A set of parameter values $\Theta = \{\theta\}$
- A set of possible solutions $D = \{d\}$
- The loss functions on the set $\Theta \times D$; $L(\theta, d)$ is defined
- The measurement result is a random quantity *X*, the distribution function of which depends on $\theta$, $P_\theta$.
- It is required to set a function that assigns to each possible measurement result a solution that has to be accepted: $\delta : X \to D$ ($\delta(x) = d$).

The function $\delta$ is set in such a way as to minimize the risk function $R(\theta, \delta)$ - the average loss under the assumption that the distribution function *X* is equal to $P_\theta$ :

$$R(\theta,\delta) = E_\theta[L(\theta,\delta(X))] = \int L(\theta,\delta(x)) p_\theta(x) dx$$

In accordance with the structure of the decision-making procedure (solutions set), it is possible to distinguish three main types of problems.

### 2.1. Quantitative estimation

In the case given, the set of possible solutions coincides with the space of parameters. The loss function in this case describes the measure of proximity of the parameter estimate $\delta(x)$ to the true value $\theta$. The most common The loss functions of the form $L(\theta, \delta(x)) = (\delta(x) - \theta)^2$ are most common. Other approaches are based on applying the maximum likelihood principle, robust, non-parametric evaluation.

In this case, the risk is equal to the dispersion of the estimate $R(\theta,\delta) = E_\theta[L(\theta,\delta(X))] = \int (\delta(x) - \theta)^2 p_\theta(x) dx = \sigma^2$. In the concept of measurement uncertainty, it is the square of the dispersion that is used as the main characteristic of uncertainty (standard uncertainty) [1].

### 2.2. Hypothesis testing

The set of solutions *D* consists of two possible decisions: to accept $(d_0)$ or reject $(d_1)$ the null hypothesis $H_0$. The null hypothesis can be simple, for example, $H_0 : \theta = \theta_0$ ($H_1 : \theta \in \Theta \setminus \theta_0$), or complex, for example, $H_0 : \theta \le \theta_0$ ($H_1 : \theta \ge \theta_0$). The set of measurement results is divided into two subsets $X = X_0 \otimes X_1$. The function $\delta(x)$ (in this statement the criterion for accepting the hypothesis) is given as follows:

$$\delta(x) = d_0, x \in X_0$$
$$\delta(x) = d_1, x \in X_1$$

The loss function has the form: $L(\theta, \delta(x)) = 0$, if the correct decision is made and $L(\theta, \delta(x)) = 1$, if an erroneous decision is made. Two types of errors are possible to reject the null hypothesis when it is true (error of the first kind) or to accept it when it is erroneous (error of the second kind). As a rule, it is assumed that the hypotheses are unequal in rights and the consequences of the wrong decision are also different.

This is reflected in the terminology: the probability of error of the first kind is called the level of significance $\alpha = P(H_1|H_0) = \int p_0(x)dx$, and the probability of rejecting the null hypothesis, provided that it is false (wrong) by the power of the criterion: $1 - \beta = P(H_1|H_1) = \int_{X_1} p_\theta(x)dx$.

The theory of statistical hypotheses testing is widely used in evaluation of the influence of measurement uncertainty on the quality of decision-making (conformity testing) in theoretical and legal metrology [2,3]. In this case, the probabilities of errors of the first and second kind are the indicators of the quality of the decisions made.

Reducing the measurement uncertainty leads to a reduction of the both errors, in this case it is not possible to talk about the uncertainty optimization. Ensuring the requirements for errors of the first and second kind is achieved by setting upper limits for uncertainty and setting the function $\delta(x)$.

### 2.3. General case - loss optimization

In the general case, the loss function has a more complicated form than that indicated in the previous items. Let's consider an example from metrological practice, namely the control of finding a parameter being measured in the given admission (a problem of the technological control). On the basis of the measurement result $\{\hat{a}, u(a)\}$ of the parameter under control, ***a***, it is required to decide not to exceed the limit set for the parameter, a≤A. .

Let's consider a simple rule of decision-making (DMR), namely, if $\hat{a} \leq A$, then a decision is made to find the parameter under control within the given boundaries, and vice versa, if $\hat{a} > A$, then the decision is made about the exit of the parameter beyond the established boundaries. Thus, the set of possible decisions consists of two elements D = $\{d_0, d_1\}$, where $d_0$ is the compliance decision, and $d_1$ is the discrepancy decision. The loss function is given on the set $R \times D$, where $R$ is the set of possible values of the parameter under control.

The loss in making an erroneous decision $L(a, d(\hat{a}))$ is defined as follows

$$L(a, d(\hat{a})) = \begin{cases} 0, & a \leq A, \ d_0 \\ c_0(a-A)^p, & a > A, \ d_0 \\ 0, & a > A, \ d_1 \\ c_1, & a \leq A, \ d_1 \end{cases}$$

The chosen form of the loss function reflects the fact that, when making the correct decision, the losses are zero, and in the case of an erroneous decision relative to finding the parameter within the limits given, the losses are proportional to the value of the parameter output beyond the upper limit. When an erroneous decision related to boundaries going beyond the boundaries takes place, the losses become fixed.

For the selected loss function, the average risks of a "consumer" and "producer" will be, respectively:

$$R(d_0|\hat{a}) = \frac{c_0}{\sqrt{2\pi}u(a)} \int_A^{\infty} (x-A)^p exp\left\{-\frac{1}{2}\frac{(x-\hat{a})^p}{u^2(a)}\right\} dx + \frac{c_0}{\sqrt{2\pi}u(a)} \int_{-\infty}^{-A} (x-A)^p exp\left\{-\frac{1}{2}\frac{(x-\hat{a})^p}{u^2(a)}\right\} dx$$

$$R(d_1|\hat{a}) = \frac{c_1}{\sqrt{2\pi}u(a)} \int_{-A}^{A} exp\left\{-\frac{1}{2}\frac{(x-\hat{a})^p}{u^2(a)}\right\} dx$$

When solving the problem of rational establishment of target uncertainty to the average losses from erroneous decisions (solutions), it is necessary to add the costs of performing measurements, which are inversely proportional to the square of the standard uncertainty [4, 5]. Thus, the target uncertainty is chosen from the condition of ensuring a minimum of one of the following expressions:

$$\frac{Q}{u^2(a)} + R(d_0|\hat{a})$$

$$\frac{Q}{u^2(a)} + R(d_1|\hat{a})$$

The difficulties in applying this approach in practice are connected with the lack of information for the justified assignment of the constants $Q$ and $c_0$. But it is just this approach that allows us to solve the problem of optimizing the uncertainty in full.

## 3. Conclusions

The measurement problem is always part of a more general problem of making decisions on the basis of measurement results. This common task forms certain requirements for the measurement procedure, which at times is difficult to formalize and express quantitatively. In the paper there is a consideration of the mathematical models describing the influence of measurement uncertainty on the quality of decisions (solutions) through the corresponding loss functions when making erroneous decisions. These models allow solving problems of optimization of measurement uncertainty provided that their parameters are set on the basis of available information

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