MEASUREMENT UNCERTAINTY EVALUATION IN NONLINEAR MODEL EQUATIONS

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Abstract: The analysis of the nonlinear model equation is carried out. The nonlinear model equation in a Taylor series is expanded. It is shown that the bias in the estimation of the total standard uncertainty in GUM is due to the terms of the expansion of the second degree. To eliminate this bias, it is necessary to take into account the kurtosis of the input quantities distributions.

A finite increments method for obtaining reliable estimates of the uncertainties contributions for nonlinear model equations is proposed.

A practical example of calculation of comparison loss in microwave power meter calibration by various methods was considered. Estimates of the numerical value and measurement uncertainty obtained with GUM in this case have a bias. The result obtained with the finite increments method coincides with the results obtained using the Monte Carlo method.

Key-Words: measurement uncertainty, nonlinear model equations, finite increments method, Monte Carlo method.

1. Introduction

In general, the model of the measurand as a function of several input quantities $X_1, X_2, ..., X_N$, can be represented in the following form [1]:

$$Y = f(X_1, X_2, ..., X_N) \ . \tag{1}$$

In the GUM [1] the value of the measurand y and its standard uncertainty $u(y)$ are calculated by the formulas:

$$y = f(x_1, x_2, ..., x_N) \ , \tag{2}$$

$$u(y) = \sqrt{\sum_{j=1}^{N} c_j^2 u_j^2 + 2\sum_{j=1}^{N-1}\sum_{i=j+1}^{N} c_j c_i \operatorname{cov}(x_j, x_i)} \ , \tag{3}$$

where u_j – the standard uncertainty of the *j*-th input quantity;

$\operatorname{cov}(x_j, x_i)$ – covariance of the *i*-th and *j*-th input quantities;

$\frac{\partial}{\partial}$ – sensitivity coefficient for the *j*-th input quantity.

In the nonlinear equation (1), the estimates (2) and (3) of y and $u(y)$ have a bias, the value of which depends on the type of the model equation and the values of the relative uncertainties of the input quantities [2].

The bias compensation of y in [1] is not carried out, and the bias compensation of $u(y)$ is realized in [1] by taking into account the terms of the second order of the expansion (1) in the Taylor series.

In this case, it is necessary to have the values of the partial second-order derivatives function (1) with respect to the input quantities $X_1, X_2, ..., X_N$.

It should be noted that the expression for an unbiased estimate of the standard uncertainty of the measurand is given in [1] only for the normal distribution law of input quantities.

The purpose of this report is to obtain expressions that provide unbiased estimate of the value and standard uncertainty of the measurand for nonlinear model equations.

2. Expansion in a Taylor series of the model equation

Expression (1) can be represented as a Taylor series expansion:

$$Y = f(x_1, x_2, ..., x_N) + \sum_{j=1}^{N} c_j (X_j - x_j) + $$

$$+ \frac{1}{2} \sum_{\substack{j=1 \\ i=1}}^{N} c_{ji} (X_j - x_j)(X_i - x_i) + R_0 \ , \tag{4}$$

in which $c_{ji} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ – is the mixed partial derivative (1) with respect to X_j and X_i;

R_0 – is the remainder term of the series.

The mathematical expectation of the expression (4) for R_0 = 0 will have the following form:

$$E(Y) = E[f(x_1, x_2, ..., x_N)] + \sum_{j=1}^{N} c_j E(X_j - x_j) + $$

$$+\frac{1}{2}\sum_{j,i=1}^{N} c_{ji} E[(X_j - x_j)(X_i - x_i)] \; . \qquad (5)$$

Taking into account that $E(X_j - x_j) = 0$, and for uncorrelated input values

$$E[(X_j - x_j)(X_i - x_i)] = \begin{cases} u_j^2, \text{ when } j = i; \\ 0, \quad \text{when } j \neq i, \end{cases}$$

we get the following expression:

$$E(Y) = y = f(x_1, x_2, ..., x_N) + \frac{1}{2}\sum_{j=1}^{N} c_{jj} u_j^2 \; , \qquad (6)$$

where $c_{jj} = \frac{\partial^2 f}{\partial x_j^2}$ – is the partial derivative of the second order of the measurand with respect to the j-th input quantity.

Thus, to reduce the bias of measurand estimate when the nonlinear model equation is used, it is necessary to take into account the standard uncertainties of input quantities.

Write down the expression for the difference between the measurand and its mathematical expectation (6):

$$Y - E(Y) = Y - f(X_1 - x_1) - \frac{1}{2}\sum_{j=1}^{N}\left(\frac{\partial^2 f}{\partial X_1^2}\right) u_j^2 =$$

$$= \sum_{j=1}^{N} c_j (X_j - x_j) + \frac{1}{2}\sum_{j=1}^{N}\left(\frac{\partial^2 f}{\partial X_1^2}\right)[(X_j - x_j)^2 - u_j^2] +$$

$$+\sum_{j=1}^{N-1}\sum_{i=j+1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i}(X_j - x_j)(X_i - x_i). \qquad (7)$$

The mathematical expectation of the square of the resulting expression will be an unbiased estimate of its variance:

$$E[Y - E(Y)]^2 = u^2(Y) = \sum_{j=1}^{N} c_j^2 u_j^2 +$$

$$+\frac{1}{4}\sum_{j=1}^{N} c_{jj}^2 (\mu_j - 1) u_j^4 + \sum_{j=2}^{N}\sum_{i=1}^{j-1} c_{ji}^2 u_j^2 u_i^2 \qquad (8)$$

where $\mu_j = E[(X_j - x_j)^4]/u_j^4$ – is the normalized fourth-order central moment of the j-th input quantity.

The values of μ_j for different distributions laws is given in Table. 1.

Table 1. The values of μ_j for different distributions laws of the input quantities

The laws of distributions	μ_j
Arcsine	1,5
Uniform	1,8
Triangular	2,4
Gaussian	3

Thus, the value of the variance of the measurand will depend on the laws of distributions of the input quantities.

So, for example, for Gaussian distribution laws of all input quantities μ_j= 3, therefore expression (8) takes the form known from the literature [3]:

$$u^2(Y) = \sum_{j=1}^{N} c_j^2 u_j^2 + \frac{1}{2}\sum_{j=1}^{N} c_{jj}^2 u_j^4 + \sum_{j=2}^{N}\sum_{i=1}^{j-1} c_{ji}^2 u_j^2 u_i^2 \; . \qquad (9)$$

For a function of single variable, the value of the standard uncertainty of the measurand takes the following form:

$$u^2(y) = c_1^2 u_1^2 + \frac{1}{4}\left(\frac{\partial^2 f}{\partial X^2}\right)^2 (\mu - 1) u_1^4 \qquad (10)$$

It is seen from expressions (8)-(10) that in order to obtain an unbiased estimate of the variance of the measurand, it is necessary to know the second partial derivatives of the measurand with respect to the corresponding input quantities, i.e. the model of the measurand should not only be fully known, but also twice differentiable.

3. The application of the finite increments method to calculate the unbiased estimate of the numerical value and the uncertainty of the measurand

The method of finite increments is based on the application of difference derivatives of different orders [4].

The difference partial derivative of the first order of the measurand with respect to the j-th input will be equal:

$$c_j^* = \frac{f[x_1,...,(x_j+u_j),...,x_N]-f[x_1,...,(x_j-u_j),...,x_N]}{2u_j} \quad (11)$$

The difference partial derivative of the second order of the measurand with respect to the j-th input quantity will be equal:

$$c_{jj}^* = \frac{1}{u_j^2}\{f[x_1,...,(x_j+u_j),...,x_N]-2f(x_1,x_2,...,x_N)+$$
$$+f[x_1,...,(x_j-u_j),...,x_N]\}. \quad (12)$$

The difference partial derivative of the second order of the measurand with respect to the j-th and i-th input quantities will be equal:

$$c_{ji}^* = \frac{1}{4u_1u_2}[f(x_1,...,x_j+u_j,...,x_i+u_i,...,x_N)-$$
$$-f(x_1,...,x_j-u_j,...,x_i+u_i,...,x_N)-$$
$$-f(x_1,...,x_j+u_j,...,x_i-u_i,...,x_N)]+$$
$$+f(x_1,...,x_j-u_j,...,x_i-u_i,...,x_N)]. \quad (13)$$

When the expressions (11), (12) in (6) are substituted, we obtain an unbiased estimate of the numerical value of the measurand [4]:

$$y^* = \sum_{j=1}^{N}\frac{f(x_1,...,x_j+u_j,...,x_N)+f(x_1,...,x_j-u_j,...,x_N)}{2}-$$
$$-(N-1)f[x_1,x_2,...,x_N]. \quad (14)$$

For a function of single variable, the value of the measurand takes the following simple form:

$$y^* = \frac{f(x_1+u_1)+f(x_1-u_1)}{2}. \quad (15)$$

For the function of N variables, the calculation of the partial difference derivatives (11) - (13) is easily realize by using a simple program executed in the MS Excel.

4. An example of calculation of comparison loss in the calibration of a microwave power meter

In subsection 9.4 of Appendix 1 to GUM [5], in the example of "Comparison loss in the calibration of a microwave power meter", the function $Y = X_1^2 + X_2^2$ for $x_1 = 0$, $x_2 = 0$ and standard uncertainties $u_1 = 0{,}005$, $u_2 = 0{,}005$ is considered. In this case we have:

y_{GUM} =0+0=0;

$$u(y_{\text{GUM}}) = \sqrt{(2x_1u_1)^2+(2x_2u_2)^2} = 0.$$

Applying the method of finite increments, we obtain:

$$y^* = \frac{[(x_1+u_1)^2+x_2^2]+[(x_1-u_1)^2+x_2^2]}{2}+$$
$$+\frac{[x_1^2+(x_2+u_2)^2]+[x_1^2+(x_2-u_2)^2]}{2}-(x_1^2+x_2^2)=$$
$$=\frac{[(0+u_1)^2+0^2]+[(0-u_1)^2+0^2]}{2}+$$
$$+\frac{[0^2+(0+u_2)^2]+[0^2+(0-u_2)^2]}{2}-0=$$
$$=u_1^2+u_2^2 = 0{,}005^2+0{,}005^2 = 0{,}00005$$

In [5] the value of y_{MCM} = 0,00005 was obtained by the Monte Carlo method, which coincides with the value of y^*.

For the calculation $u(y^*)$, we find the values of the partial derivatives (11) - (13):

$$c_1^* = \frac{1}{2u_1}\{[(x_1+u_1)^2+x_2^2]-[(x_1-u_1)^2+x_2^2]\} = 0;$$

$$c_2^* = \frac{1}{2u_2}\{[x_1^2+(x_2+u_2)^2]-[x_1^2+(x_2-u_2)^2]\} = 0;$$

$$c_{11}^* = \frac{[(x_1+u_1)^2+x_2^2]-2(x_1^2+x_2^2)+[(x_1-u_1)^2+x_2^2]}{u_j^2} = 2;$$

$$c_{22}^* = \frac{[x_1^2+(x_2+u_2)^2]-2(x_1^2+x_2^2)+[x_1^2+(x_2-u_2)^2]}{u_i^2} = 2;$$

$$c_{12}^* = \frac{1}{4u_1u_2}\{[(x_1+u_1)^2+(x_2+u_2)^2]-$$
$$-[(x_1-u_1)^2+(x_2+u_2)^2]-[(x_1+u_1)^2+(x_2-u_2)^2]+$$
$$+[(x_1-u_1)^2+(x_2-u_2)^2] = 0.$$

Therefore, for Gaussian distributed input quantities, we have:

$$u(y^*) = \sqrt{c_1^{*2}u_1^2 + c_2^{*2}u_2^2 + \frac{c_{11}^{*2}u_1^4 + c_{22}^{*2}u_2^4}{2} + c_{12}^{*2}u_1^2u_2^2} =$$

$$= \sqrt{\frac{4}{2}(u_1^4 + u_2^4)} = \sqrt{4 \cdot 0{,}005^4} = 0{,}00005.$$

In [4], a value of $u(y_{\mathrm{MCM}})$ =0,00005 was obtained by the Monte Carlo method, which coincides with the value $u(y^*)$.

5. Conclusions

Thus, the proposed approach makes it possible to reduce the bias in the estimates of measurand and its uncertainty for nonlinear model equations.

References

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